Nonlinear operators on graphs via stacks

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Abstract. We consider a framework for nonlinear operators on functions evaluated on graphs via stacks of level sets. We investigate family of transformations which includes adaptive flat and non-flat erosions and dilations. Additionally, we note the connection to mean motion curvature on graphs. Proposed operators are illustrated in the cases of functions on graphs, textured meshes and graphs of images.

1 Introduction

Recent years have witnessed an enormous growth of interest in the description and analysis of problems via similarities or dependencies between data elements. A common way to represent this structure is to use graphs, so that data elements are indexed by graph nodes, and the strength of dependences between pairs of elements is represented by corresponding weighted graph edges. In this paper, we analyse nonlinear (morphological) operators in the context of *discrete signal processing on graphs* [1]. Our framework is used to extend the traditional adaptive (non-flat) morphology on images to more complex structures as sets of images, meshes, point clouds [2] and so on. In graph-based modelling, digital images are a particular case, where the pixel information is represented by the two-dimensional rectangular grid, and pixels correspond to graph nodes related by links according to the four or eight adjacent neighbourhood.

On the one hand, we note that in the literature, one can find some works about nonlinear filters on graphs and hypergraphs, particularly mathematical morphology operators in the algebraic sense [3,4,5,6,7,8,9], where the couple of nonlinear operator (dilation/erosion) are maps from two different lattices, i.e., they are maps "from nodes to edges" or "edges to nodes". On the other hand, some regularisation techniques and nonlinear operators have been introduced on function evaluated on graph via directional derivative [10] [11] [12] or discrete version of the p-Laplacian [13]. We adopt a different viewpoint, our approach is inspired from the signal processing approach on graphs [14,15,16,17]. Thus, we firstly review graph signal decomposition by upper-level sets, convolution and diffusion on graphs, and then we present a general formulation of flat and non-flat morphology on graphs, a family of nonlinear transformations and its connection to mean curvature motion on graphs [18,17,19]. Finally, the experimental section includes some examples to illustrate the interest of our method.

2 Convolution and morphology by stacks on graphs

2.1 Notation

We start by introducing the notation used throughout this paper. The objects under study are consider as the nodes of the graph $\mathcal{G}$. A simple, connected, undirected, and weighted graph $\mathcal{G} = (\mathcal{V}, E)$ consists of a set of nodes $\mathcal{V} = \{1, 2, \ldots, N\}$ and edges $E = \{(n, m, w_{nm})\}, n, m \in \mathcal{V}$, where (n, m, w_{nm}) denoted an edge of weight w_{nm} between node n and m. The degree d_n of a node n is the sum of the edge-weights connected to node n, and the degree matrix of the graph consists of degrees of all nodes arranged in a diagonal matrix $\mathbf{D} = diag\{d_1, d_2, \ldots, d_N\}$. Denote the maximal and minimal degrees by $d_+ = \max_{i \in \mathcal{V}} d_i$ and $d_- = \min_{i \in \mathcal{V}} d_i$. The adjacency matrix $\mathbf{W}$ of the graph is an $N \times N$ matrix with $\mathbf{W}(n, m) = w_{nm}$, the *combinatorial Laplacian* matrix is $\mathbf{L} = \mathbf{D} - \mathbf{W}$ and the *graph Laplacian* $\mathcal{L} = \mathbf{I} - \mathbf{D}^{-1/2}\mathbf{W}\mathbf{D}^{-1/2}$ is a generalisations of the Laplacian on the grid, where frequency and smoothness are relative to $\mathbf{W}$ and interrelated though these operators [20]. A graph signal is defined as a scalar valued discrete mapping $\mathbf{f} : \mathcal{V} \to \mathbb{R}$, such that $\mathbf{f}(n)$ is the value of the signal on node n. Thus a graph signal can also be represented as a vector $\mathbf{f}$ in the space of functions from $\mathcal{V}$ to $\mathbb{R}$, denoted by V, with indices corresponding to the nodes in the graph. Additionally, we often analyse operators transforming signals evaluated on graphs, for instance $\phi : \mathbf{f} \to \phi(\mathbf{f})$, in this case we say that $\phi \in \mathsf{V} \times \mathsf{V}$. Finally, the *graph Fourier transform* $\hat{\mathbf{f}}$ of a function $\mathbf{f} \in \mathsf{V}$ is the expansion of $\mathbf{f}$ in terms of the eigenvectors of the graph Laplacian, denoted by Λ_l, with $l = 1, \cdots, N$. More precisely, it is defined by $\hat{\mathbf{f}}(l) := \langle \mathbf{f}, \Lambda_l \rangle = \sum_{n=1}^{N} \Lambda_l^*(n)\mathbf{f}(n)$, where according to [14] adopts the conjugate-linear in the definition.[1]

Definition 1. *The* upper level set *(ULS) of* $\mathbf{f} \in \mathsf{V}$ *with level* $\lambda \in \mathbb{R}$ *is defined by*

$$\chi(\mathbf{f}, \lambda) = \{v \in \mathcal{V} : \mathbf{f}(v) \geq \lambda\}.$$

The set of ULS constitutes a family of decreasing sets: $\lambda \geq \mu \Rightarrow \chi(\mathbf{f}, \lambda) \subseteq \chi(\mathbf{f}, \mu)$ and $\chi(\mathbf{f}, \lambda) = \cap\{\chi(\mathbf{f}, \mu), \mu < \lambda\}$. Any graph signal $\mathbf{f} \in \mathsf{V}$ can be viewed as a unique stack of its cross-sections, which leads to the following superposition description.

Definition 2. *The* threshold-max superposition *of* $\mathbf{f} \in \mathsf{V}$ *is defined by:*

$$\mathbf{f}(n) = \bigvee_{\lambda \in \mathbb{R}} \{n \in \chi(\mathbf{f}, \lambda)\}. \tag{1}$$

Similar to the image description as a topographic surface in [21,22,23], we consider here the alternative stacking reconstruction using a numerical sum of the characteristic function of upper level sets.

[1] For a given matrix $\mathbf{W} = [\mathbf{W}(i, j)]$, the *conjugate* of $\mathbf{W}$ to be $\mathbf{W}^* = [-\mathbf{W}(j, i)]$, i.e., $\mathbf{W}^*$ is derived from $\mathbf{W}$ by transposing and negating.

Definition 3. *The* threshold-linear superposition *of* $\mathbf{f} \in \mathsf{V}$ *is defined by:*

$$\mathbf{f}(n) = \int_0^{+\infty} \chi(\mathbf{f}, \lambda)(n) d\lambda \tag{2}$$

In the particular case of discrete range, $\mathcal{T} = \{c_1, c_2, \ldots, c_{|\mathcal{T}|}\}$, the signal $\mathbf{f}$ can be reconstructed from the discrete stack $\chi(\mathbf{f}, \lambda)$ via addition, i.e., $\mathbf{f}(n) = \sum_{\lambda \in \mathcal{T}} \chi(\mathbf{f}, \lambda)(n)$.

Definition 4. *We shall say that an operator* $\phi \in \mathsf{V} \times \mathsf{V}$ commutes with thresholding *if*

$$\phi\left(\chi(\mathbf{f}, \lambda)\right) = \chi\left(\phi(\mathbf{f}), \lambda\right) \tag{3}$$

for any signal $\mathbf{f} \in \mathsf{V}$ *and any value* $\lambda \in \mathbb{R}$.

In other words, if an operator ϕ commutes with thresholding, processing by ϕ the upper level set at λ gives the same result as processing first by ϕ the signal $\mathbf{f}$ and then thresholding $\phi(\mathbf{f})$ at level λ.

Definition 5. *We shall say that an operator* ϕ *obeys the threshold-linear superposition provided that*

$$\phi(\mathbf{f}) = \int_0^{+\infty} \phi\left(\chi(\mathbf{f}, \lambda)\right) d\lambda \tag{4}$$

for any signal $\mathbf{f} \in \mathcal{V} \times \mathbb{R}^+$. *Respectively,* $\sum_{\lambda \in \mathcal{T}^+} \phi\left(\chi(\mathbf{f}, \lambda)\right)$ *for positive discrete signals.*

As it was pointed out in [23], for grey scale images, the threshold-max superposition in (1) is more general than the thresholded-sum superposition in (4) since the latter applies only to nonnegative input signals, while the former applies to any real-valued input signal. But alternatively, the max-superposition restricts the class of operators since it requires the result of $\phi\left(\chi(\mathbf{f}, \lambda)\right)$ that are binary signals, an assumption not needed by the sum-superposition. In fact, the threshold sum/integral ties well also with linear systems. In our case, we assume that $\mathbf{f} \in \mathcal{V} \times \mathbb{R}^+$ is continuous and nonnegative so we will consider the reconstruction formula given by (4).

Proposition 1. *The class of operators* $\phi \in \mathsf{V} \times \mathsf{V}$ *that obey the threshold-linear superposition: a) is closed under minimum, maximum and composition. b) It forms a* vector space *over the field of real numbers under vector addition* $(\phi_1 + \phi_2)(\mathbf{f}) := \phi_1(\mathbf{f}) + \phi_2(\mathbf{f})$ *and the scalar multiplication* $(c\phi)(\mathbf{f}) := c\phi(\mathbf{f})$ *with* $c \in \mathbb{R}$.

2.2 Convolution on graphs

For signals $\mathbf{f}, \mathbf{g} \in L^2(\mathbb{R})$, the convolution product $\mathbf{h} = \mathbf{f} * \mathbf{g}$ satisfies

$$\mathbf{h} = (\mathbf{f} * \mathbf{g}) = \int_{\mathbb{R}} \hat{\mathbf{h}}(\xi) \exp\{2\pi i \xi t\} d\xi = \int_{\mathbb{R}} \hat{\mathbf{f}}(\xi) \hat{\mathbf{g}}(\xi) \exp\{2\pi i \xi t\} d\xi. \tag{5}$$

By replacing the complex exponentials in (5) with the graph Fourier transform, i.e., the graph Laplacian eigenvectors Λ_l, in [14] has defined a *generalised convolution* of signals $\mathbf{f}, \mathbf{g} \in \mathsf{V}$ by

$$(\mathbf{f} * \mathbf{g}) := \sum_{l=1}^{N} \left[\hat{\mathbf{f}}(l) \hat{\mathbf{g}}(l) \Lambda_l \right] = \sum_{l=1}^{N} \left[\left(\sum_{k=1}^{N} \Lambda_l^*(k) \mathbf{f}(k) \right) \left(\sum_{k=1}^{N} \Lambda_l^*(k) \mathbf{g}(k) \right) \Lambda_l \right]$$
$$\sum_{l=1}^{N} \left[\left(\Lambda_l^* \mathbf{f} \right) \left(\Lambda_l^* \mathbf{g} \right) \Lambda_l \right] = \sum_{l=1}^{N} \left[\Lambda_l^* (\mathbf{f}\mathbf{g}) \Lambda_l \right] = \sum_{l=1}^{N} \mathbf{f}(l) \mathbf{g}(l).$$

Proposition 2. *The linear operator associated to the convolution signal function* $\mathbf{g}$ *commutes with the stacking of cross-sections according to the threshold-linear superposition,i.e.,*

$$(\mathbf{f} * \mathbf{g}) = \int_0^{+\infty} \left(\chi(\mathbf{f}, \lambda) * \mathbf{g} \right) d\lambda \tag{6}$$

2.3 Diffusion on graphs

Consider an arbitrary graph $\mathcal{G} = (\mathcal{V}, E, \mathbf{W})$ with Laplacian matrix $\mathbf{L}$ and a signal $\mathbf{f} \in \mathcal{V} \to \mathbb{R}^N$. For a given constant $\sigma > 0$, define the time-varying vector $\mathbf{f}_{\sigma,t} \in \mathbb{R}^N$ as the solution of the linear differential equation:

$$\frac{\partial \mathbf{f}_{\sigma,t}}{\partial t} = -\sigma \mathbf{L} \mathbf{f}_{\sigma,t}, \quad \mathbf{f}_0 = \mathbf{f}, \tag{7}$$

where σ is the thermal conductivity [16] and controls the heat diffusion rate. The differential equation in (7) represents the heat diffusion process on the graph $\mathcal{G}$ due to the fact that $-\mathbf{L}$ can be shown to be the discrete approximation of the continuous Laplacian operator used to characterise the heat diffusion in physics [16,24]. The general solution of the heat equation on $\mathbb{R}^N$ is obtained by convolution [25]. However, the solution of (7) denoted by $\mathbf{f}^{\mathbf{L}}_{\sigma,t} \in \mathsf{V} \times \mathsf{V}$, is given by the matrix exponential as follows

$$\mathbf{f}^{\mathbf{L}}_{\sigma,t} := \exp\left(-\sigma \mathbf{L} t\right) \mathbf{f} \tag{8}$$

which can be verified by direct substitution in (7). It is important to note that for a given time t, the n-th element of $\mathbf{f}_{\sigma,t}(n)$ is

$$\frac{\partial \mathbf{f}^{\mathbf{L}}_{\sigma,t}(n)}{\partial t} = \sum_{k \in \mathcal{N}(n)} \sigma \mathbf{W}(n,k) (\mathbf{f}^{\mathbf{L}}_{\sigma,t}(k) - \mathbf{f}^{\mathbf{L}}_{\sigma,t}(n))$$

where $\mathcal{N}(n)$ is the neighbourhood of n, i.e., the k such that $\mathbf{W}(i,k) > 0$. Thus, the heat flow on an edge grows proportionally with both the "temperature differential" $\mathbf{f}^{\mathbf{L}}_{\sigma,t}(k) - \mathbf{f}^{\mathbf{L}}_{\sigma,t}(n)$ and the weight $\mathbf{W}(n,k)$. Now in Prop. 3, we see the behaviour of the graph diffusion in the stack of cross-sections.

Proposition 3. *The operator* $\mathbf{f}^{\mathbf{L}}_{\sigma,t}$ *in eq.* (8) *associated to the diffusion of a graph signal* $\mathbf{f}$*, commutes with the stacking of cross-sections according to the threshold-linear superposition, i.e.,*

$$\mathbf{f}^{\mathbf{L}}_{\sigma,t}(n) = \int_0^{+\infty} (\chi(\mathbf{f},\lambda))^{\mathbf{L}}_{\sigma,t}(n) d\lambda \tag{9}$$

Note that the right part of the equality means that the graph diffusion is applied in each upper level set of the graph signal function $\mathbf{f}$. The proof of Prop. 3 is straightforward by means of Taylor series expansion of the graph heat equation and interchanging summation and integration.

Proof.

$$\mathbf{f}^{\mathbf{L}}_{\sigma,t}(n) = \sum_{k=0}^{\infty} \frac{(-\sigma \mathbf{L} t)^k \mathbf{f}}{k!} = \int_0^{+\infty} \sum_{k=0}^{\infty} \frac{(-\sigma \mathbf{L} t)^k}{k!} \chi(\mathbf{f},\lambda)(n) d\lambda = \int_0^{+\infty} (\chi(\mathbf{f},\lambda))^{\mathbf{L}}_{\sigma,t}(n) d\lambda.$$

At this point, we should highlight that the behaviour of the diffusion in (8) is controlled by the choice of the Laplacian matrix, i.e, therefore expression (8) includes isotropic and anisotropic diffusion.

2.4 Morphological operators in graphs

In the case of a graph value function $\mathbf{f} \in \mathsf{V}$, we can have the following counterparts of dilation and erosion of numerical functions [26,27,28] viewed as a convolution in max-plus algebra (and its adjoint/dual algebra).

Definition 6. *The matrix* $\mathbf{W}$ *is a morphological weight matrix if* $-\infty \leq \mathbf{W}(n,m) \leq 0$*, for all* n, m.

It is important to note, that it is a really simple characterisation of the weight matrix because we do not require symmetry ($\mathbf{W}(n,m) \neq \mathbf{W}(m,n)$) neither zero-diagonal ($\mathbf{W}(i,i) \neq 0$).

Definition 7. *The dilation of a signal function* $\mathbf{f}$ *on a graph* $\mathcal{G} = (\mathcal{V}, E)$ *is defined by*

$$\delta_{\mathbf{W}}(\mathbf{f})(n) = \bigvee_{m=1}^{N} (\mathbf{f}(m) + \mathbf{W}(n,m) := \mathbf{W} \oplus \mathbf{f}(n) \tag{10}$$

and the dual adjoint erosion is given by

$$\varepsilon_{\mathbf{W}} \mathbf{f}(n) := \bigwedge_{m=1}^{N} (\mathbf{f}(m) + \mathbf{W}^*(n,m)) = \bigwedge_{m=1}^{N} (\mathbf{f}(m) - \mathbf{W}(m,n)) := \mathbf{W}^* \ominus \mathbf{f}(n) \tag{11}$$

We remark that, $\varepsilon_{\mathbf{W}}, \delta_{\mathbf{W}}$ are both in $\mathsf{V} \times \mathsf{V}$ and include morphological transformations by flat, non-flat [26,27], adaptive [28] and nonlocal structuring elements [29]. The following proposition list include some properties easy to check.

Proposition 4. *Let* $\mathbf{f}_i \in \mathsf{V}, i = 1, \ldots, k$ *be signal on graph and let* $\mathbf{W}_j, j = 1, \ldots m$ *be morphological weights matrices:*

1. $\mathbf{W}_m \oplus \ldots \oplus \mathbf{W}_1 \oplus (\mathbf{f}) = (\sum_{j=1}^{m} \mathbf{W}_j) \oplus \mathbf{f}$
2. $\mathbf{W} \oplus (\bigvee_{i=1}^{k} \mathbf{f}_i) = \bigvee_{i=1}^{k} (\mathbf{W} \oplus \mathbf{f}_i)$
3. $\mathbf{W}_m^* \ominus \ldots \ominus \mathbf{W}_1^* \ominus \mathbf{f} = (\sum_{j=1}^{m} \mathbf{W}_j^*) \ominus \mathbf{f}$
4. $\mathbf{W}^* \ominus (\bigwedge_{i=1}^{k} \mathbf{f}_i) = \bigwedge_{i=1}^{k} (\mathbf{W}^* \ominus \mathbf{f}_i)$
5. *For* $\mathbf{f} \leq \mathbf{g}$ *then* $\mathbf{W} \oplus \mathbf{f} \leq \mathbf{W} \oplus \mathbf{g}$
6. *For* $\mathbf{f} \leq \mathbf{g}$ *then* $\mathbf{W}^* \ominus \mathbf{f} \leq \mathbf{W}^* \ominus \mathbf{g}$
7. $\mathbf{W}^* \ominus (\mathbf{W} \oplus \mathbf{f}) \geq \mathbf{f} \geq \mathbf{W} \oplus (\mathbf{W}^* \ominus \mathbf{f})$

A crucial point is the existence of a *Galois adjunction theorem* [30,31] for graph valued signals.

Theorem 1. *Given a* $\mathbf{W}$ *morphological weight matrix and, the pair of operators* $(\varepsilon_{\mathbf{W}}, \delta_{\mathbf{W}})$*, defines an* Galois adjunction*, i.e., for all* $\mathbf{f}, \mathbf{g} \in \mathsf{V}$*, we have* $\mathbf{W} \oplus \mathbf{f} \leq \mathbf{g} \iff \mathbf{f} \leq \mathbf{W}^* \ominus \mathbf{g}$.

Proof.

$$\mathbf{W} \oplus \mathbf{f} \leq \mathbf{g} \iff \forall n, \quad \bigvee_{m=1}^{N} (\mathbf{f}(m) + \mathbf{W}(n,m)) \leq \mathbf{g}(n)$$

$$\iff \forall n, \forall m, \mathbf{f}(m) + \mathbf{W}(n,m) \leq \mathbf{g}(n), \iff \forall n, \forall m, \mathbf{f}(m) \leq \mathbf{g}(n) - \mathbf{W}(n,m),$$

$$\iff \forall m, \mathbf{f}(m) \leq \bigwedge_{n=1}^{N} \mathbf{g}(n) - \mathbf{W}(n,m), \iff \forall m, \mathbf{f}(m) \leq \bigwedge_{n=1}^{N} \mathbf{g}(n) + \mathbf{W}^*(m,n),$$

$$\iff \forall m, \mathbf{f} \leq \mathbf{W}^* \ominus \mathbf{g}.$$

However, we do not have an order between the original signal $\mathbf{f}$ and its dilation or erosion, *i.e.*, we have $\mathbf{W}^* \ominus \mathbf{f} \leq \mathbf{W} \oplus \mathbf{f}$, but $\mathbf{f} \not\leq \mathbf{W} \oplus \mathbf{f}$ neither $\mathbf{W} \ominus \not\leq \mathbf{f}$.

Definition 8. *A morphological weight matrix* $\mathbf{W}$ *is called* conservative *if* $\mathbf{W}(i,i) = 0$ *for all* $i \in 1 \ldots, N$.

Proposition 5. *If* $\mathbf{W}$ *is an extensive morphological weight matrix then* $\mathbf{W}^* \ominus \mathbf{f} \leq \mathbf{f} \leq \mathbf{W} \oplus \mathbf{f}$ *for every* $\mathbf{f}$.

Thanks to theorem 1, we can have a large set of morphological filters such as openings, closings, alternate sequential filters, levelling and so on, because they are are defined by combination of dilations and erosions [32,33].

Definition 9. *A morphological weight matrix* $\mathbf{B}$ *is called* flat *if it is morphological weight matrix and* $\mathbf{B}(i,j) = 0$ *or* $\mathbf{B}(i,j) = -\infty$ *for all* $i, j \in 1 \ldots, N$.

Proposition 6. *The flat dilation and erosion obey both the threshold-max superposition and the threshold linear superposition, i.e.,*

$$\delta_{\mathbf{B}}(\mathbf{f}) = \bigvee_{\lambda=0}^{+\infty} \{\delta_{\mathbf{B}}\left(\chi(\mathbf{f},\lambda)\right) = 1\} = \int_0^{+\infty} \delta_{\mathbf{B}}(\chi(\mathbf{f},\lambda))d\lambda$$
$$\varepsilon_{\mathbf{B}}(\mathbf{f}) = \bigvee_{\lambda=0}^{+\infty} \{\varepsilon_{\mathbf{B}}\left(\chi(\mathbf{f},\lambda)\right) = 1\} = \int_0^{+\infty} \varepsilon_{\mathbf{B}}(\chi(\mathbf{f},\lambda))d\lambda$$

Since by Props. 1 and 6, we can directly have that the class of operators $\phi \in \mathsf{V} \times \mathsf{V}$ that obey the threshold-linear superposition also contain morphological gradients (difference between dilation and erosion), opening and closing (composition of dilation and erosion), top-hat transformation, granulometries, reconstruction operators, levelling, additive morphological decompositions [34] and skeleton transformation (based on generalised Lantuejoul formula) [35].

2.5 Morphological operators via convolution on graph

Definition 10. *For a graph signal value* $\mathbf{f} \in \mathsf{V}$*, the convolution-thresholding nonlinear operator associated to the heat diffusion* $\mathbf{W}$ *of conductivity* σ *at scale* t*, and the threshold* τ*, with* $\tau \in [0,1]$*, is the mapping* $\mathcal{F}(\mathcal{V},\mathbb{R}) \to \mathcal{F}(\mathcal{V},\mathbb{R})$ *defined by*

$$\psi_{\mathbf{L},\,\tau}(\mathbf{f}) = \int_0^{+\infty} \left[(\chi(\mathbf{f},\lambda))_{\sigma,t}^{\mathbf{L}} \geq \tau\right] d\lambda \tag{12}$$

The next proposition is easy to proof.

Proposition 7. *For all* $\mathbf{f} \in \mathsf{V}$*: a)* $\psi_{\mathbf{L},\,\tau}(\mathbf{f})$ *satisfies the threshold-linear superposition in Def. 2. b)* $\psi_{\mathbf{L},\,\tau}(\mathbf{f})$ *is monotonous with respect to the choice of* τ*, i.e,* $\tau_1 \leq \tau_2 \Rightarrow \psi_{\mathbf{L},\,\tau_1}(\mathbf{f}) \leq \psi_{\mathbf{L},\,\tau_2}(\mathbf{f})$.

We can also prove that (12) is increasing.

Proposition 8. *If* $\mathbf{f}_1 \leq \mathbf{f}_2$*, then* $\psi_{\mathbf{L},\,\tau}(\mathbf{f}_1) \leq \psi_{\mathbf{L},\,\tau}(\mathbf{f}_2)$ *for all* $\sigma, t \geq 0$.

Proof. We note that (8) follows the called *comparison principle*, (Lemma 2.6, property (d) in [17]), i.e. if $\mathbf{f}_1 \leq \mathbf{f}_2$, then $(\mathbf{f}_1)_{\sigma,t}^{\mathbf{L}} \leq (\mathbf{f}_2)_{\sigma,t}^{\mathbf{L}}$ for all $\sigma, t \geq 0$. The proposition is proved by applying this result in each upper level set and integrating in λ.

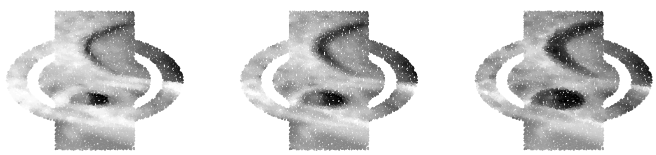
Proposition 9. *For the case of a flat morphological weight matrix* $\mathbf{B}$*, the morphological flat operators in Prop. 6 correspond to the convolution-thresholding nonlinear operator in* (12) *with particular values of* τ *as follows:* $\delta_{\mathbf{B}}(\mathbf{f}) = \lim_{\tau \to 0^+} \psi_{\mathbf{B},\,\tau}(\mathbf{f})$ *and* $\varepsilon_{\mathbf{B}}(\mathbf{f}) = \lim_{\tau \to 1^-} \psi_{\mathbf{B},\,\tau}(\mathbf{f})$.

Proposition 10. *For a binary signal* $S \in \mathcal{V}\times\{0,1\}$*, the set of measures* $Vol(S) = \sum_{i\in S} d_i$*. Let* $\rho(\mathbf{L})$ *be the spectral radius of the graph Laplacian,* $\mathbf{L}$*, then iterations in Prop. 9 on the graph with initial set* S *are stationary if either of the two conditions are satisfied:*

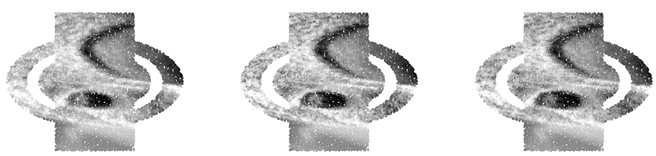
$$\sigma \leq \rho(\mathbf{L})^{-1} \log\left(1 + \tau d_{-}^{r/2} (Vol(S))^{-1/2}\right) \ \textit{or}\ \sigma \leq \frac{\tau}{||\mathbf{L}\chi(S)||_{\mathsf{V},\infty}}$$

The proof is direct by using Lemma 2.2 and Theorem 4.2 in [17].

(a) Original (b) Diffusion with ($t = 10$, $\sigma = .02$) (c) Non-flat erosion (d) Non-flat dilation



(e) τ =0.2 (f) τ =0.5 (g) τ =0.8



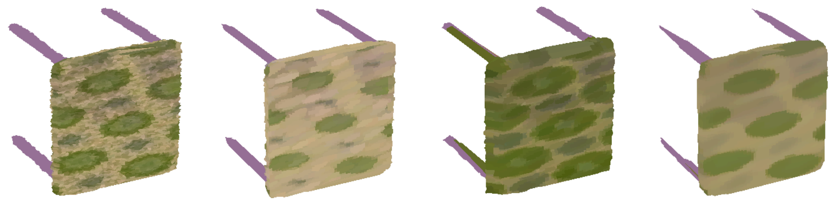
(h) τ =0.2 (i) τ =0.5 (j) τ =0.8

Fig. 1. Family of linear (b) and nonlinear transformations via thresholding of diffusion process. Second row: ($\sigma = .1, t = 10$). Third row: ($\sigma = .01, t = 10$)

Now, we point out a link of the operator in (12) with motion by mean curvature on a graph.

Proposition 11. *The operator in* (12) *in the case of* $\tau = .5$ *is an iteration with of the approximate motion by mean curvature on a graph.*

Firstly, the $\psi_{\mathbf{L},\,.5}$ is an iteration of the well-known Merriman, Bence and Osher (MBO) [37][38] threshold dynamics algorithms on graphs. The *MBO algorithm* is obtained by time splitting the *Allen-Cahn phase-field equation* for motion



(a) Original (b) Dilation (c) Erosion (d) Curvature motion

Fig. 2. Illustration of nonlinear filters on textured mesh. The textured mesh is obtained from [36]. Note that both colours and mesh coordinates change in the processing. (d) Curvature motion is obtained by iterating $\psi_{\mathbf{L},.5}$.

by mean curvature[2]. The resulting scheme alternated two steps, diffusion, and simple thresholding [17]. Secondly, several papers use the MBO algorithm on a graph to approximate motion by mean curvature [17,40]. It is important to note that the curvature is defined by means of the isotropic total variation [17] instead of the one-Laplacian as it is the case in [10,11,12,13].

3 Experimental section

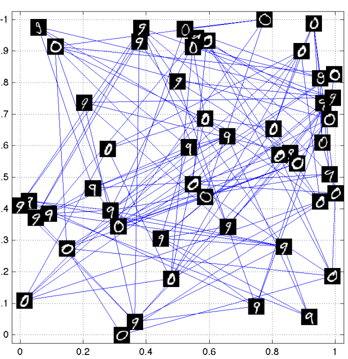
The family of filters proposed can be used to analyse any function defined on the vertices of a graph. We provide some illustrations of the results in grey scale functions on graph in Fig. 1, colour data on mesh in Fig.2 and images on a graph of images in Fig. 3. We first constructed a graph from these datasets by treating the nodes in the graph to be the sample points in the dataset and the edges weight to be the similarity between the features of the different samples. Edge weights were determined via the Radial Bases Function (RBF) kernel with σ^2 set to the variance in the respective dataset $\mathbf{W}(i,j) = \exp(-\frac{||\mathbf{x}_i - \mathbf{x}_j||^2}{2\sigma^2})$. Finally, in Fig. 3 we have considered a subset of the USPS handwritten digit database for illustration. Each image is a digit in a 28×28 grey scale image which is considered in our approach as a multivariate vector in $\mathbb{R}^{784}$. This random weight

[2] The semilinear heat equation called the *Allen-Cahn equation* is a reaction-diffusion equation of mathematical physics of the form:

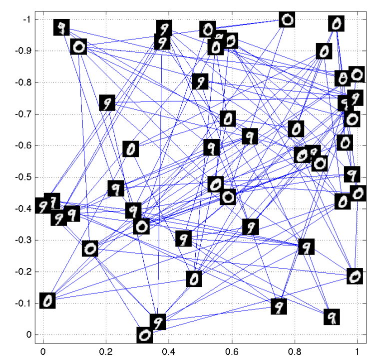
$$\frac{\partial u}{\partial t} - \delta u + \frac{W'(u)}{\epsilon^2} = 0 \in \mathbb{R}^N \times (0, \infty) \tag{13}$$

which was introduced by S.M. Allen and J.W. Can (1979) [39] to describe the process of phase separation in iron alloys, including order-disorder transitions. Here W is a function that has only two equal minima; its typical form is $W(v) = \frac{(v^2-1)^2}{2}$, W' denotes the derivative and ϵ is a positive parameter.

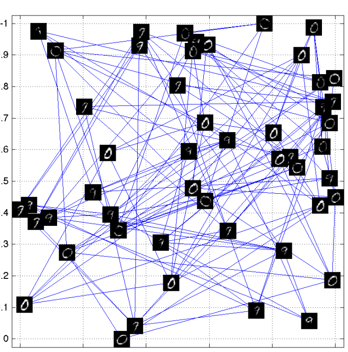
graph has nodes on images and weights in the 5-KNN graph considering the Euclidean distance. We only use 200 images of two digits (0,9) to illustrated the merits of our approach and its potential applications.



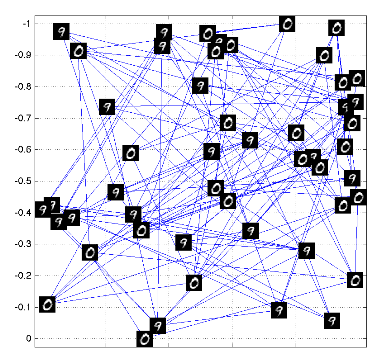
(a) Original



(b) $\tau = .9$



(c) $\tau = .1$



(d) $\tau = .5$ after 10 iterations.

Fig. 3. $\mathcal{G}$ is the 5 nearest neighbours with $\mathbf{W}$ the Euclidean distance between pairs of images. Lines are linking images that where $\mathbf{W}(i, j)$ is not zero. The parameter in the diffusion are $t = 20$ and $\sigma = .03$. Note that digits in the same class tend to be similar in (d).

4 Conclusion and perspectives

We have analysed nonlinear operators on stack of graphs as a discrete adaptation of non-flat morphological transformation. This approach is based on connection between diffusion+ thresholding operators and morphological operators. We have proved adjunction of pair dilation and erosion in signal graphs in a general case. Finally, we have illustrated the interest and behaviour of such operators in some problems of image processing and pattern recognition.

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